



TITLE:

# ON THE ARGUMENT INEQUALITY OF ANALUYTIC FUNCTIONS (Inequalities in Univalent Function Theory and Its Applications)

AUTHOR(S):

Nunokawa, Mamoru; Takahashi, Norihiro; Ogino,  
Akihisa

---

CITATION:

Nunokawa, Mamoru ...[et al]. ON THE ARGUMENT INEQUALITY OF ANALUYTIC FUNCTIONS (Inequalities in Univalent Function Theory and Its Applications). 数理解析研究所講究録 2002, 1276: 65-68

ISSUE DATE:

2002-07

URL:

<http://hdl.handle.net/2433/42303>

RIGHT:

# ON THE ARGUMENT INEQUALITY OF ANALYTIC FUNCTIONS

MAMORU NUNOKAWA [布川 護], NORIHIRO TAKAHASHI [高橋 典宏]  
AND AKIHISA OGINO [荻野 明久] (群馬大学)

ABSTRACT. Let  $p(z)$  be analytic in  $|z| < 1$ ,  $p(0) = 1$ ,  $p(z) \neq 0$  in  $|z| < 1$  and  $|\arg p'(z)| < \pi(\alpha - 1)/4$  in  $|z| < 1$  where  $1 < \alpha < 2$ . Then we have

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{in } |z| < 1.$$

## 1. INTRODUCTION.

Let  $\mathcal{N}$  be the class of all functions  $p(z)$  which are analytic in the unit disc  $\mathbb{E} = \{z : |z| < 1\}$  and equal to 1 at  $z = 0$ . We say  $p(z) \in \mathcal{N}$  a Carathéodory function if it satisfies the condition  $\operatorname{Re} p(z) > 0$  in  $\mathbb{E}$ .

If  $F(z)$  and  $G(z)$  are analytic in  $\mathbb{E}$ , then  $F(z)$  is subordinate to  $G(z)$ , written by  $F(z) \prec G(z)$ , if  $G(z)$  is univalent in  $\mathbb{E}$ ,  $F(0) = G(0)$  and  $F(z) \subset G(z)$ .

In [1, Theorem 5], Miller and Mocanu proved the following theorem.

**Theorem A.** *Let  $p(z) \in \mathcal{N}$  and suppose that*

$$p(z) + zp'(z) \prec \left[ \frac{1+z}{1-z} \right]^\alpha \implies p(z) \prec \left[ \frac{1+z}{1-z} \right]^\beta$$

where  $\alpha = \alpha(\beta) = \beta + (2/\pi)\tan^{-1}\beta$ ,  $0 < \beta < \beta_0 = 1.21872 \dots$  and  $\beta_0$  is the root of the equation

$$\beta\pi = \frac{3}{2}\pi - \tan^{-1}\beta.$$

On the other hand, Nunokawa [2] proved the following lemma.

**Lemma 1.** *Let  $p(z) \in \mathcal{N}$ ,  $p(z) \neq 0$  in  $\mathbb{E}$  and suppose that there exists a point  $z_0 \in \mathbb{E}$  such that*

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{in } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha$$

where  $0 < \alpha$ . Then we have

---

2000 Mathematics Subject Classification: Primary 30C45.

Keyword: Argument inequality.

M. NUNOKAWA, N. TAKAHASHI AND A. OGINO

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha$$

where

$$k \geq \frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg p(z_0) = \frac{\pi}{2} \alpha$$

and

$$k \leq -\frac{1}{2} \left( a + \frac{1}{a} \right) \quad \text{when } \arg p(z_0) = -\frac{\pi}{2} \alpha$$

where

$$p(z_0)^{1/\alpha} = \pm ia \quad \text{and} \quad a > 0.$$

Applying Lemma 1, we can easily obtain the following result.

**Theorem B.** Let  $p(z) \in \mathcal{N}$ ,  $p(z) \neq \mathbb{E}$  and suppose that

$$|\arg(p(z) + zp'(z))| < \frac{\pi}{2} \left( \beta + \frac{2}{\pi} \tan^{-1} \beta \right) \quad \text{in } \mathbb{E}$$

where  $0 < \beta$ . Then we have

$$|\arg p(z)| < \frac{\pi}{2} \beta \quad \text{in } \mathbb{E}.$$

**Remark 1.** For the case  $0 < \beta < \beta_0$ , Theorem B is obtained from Theorem A but Theorem B holds to be true for all the case  $0 < \beta$  if we consider the function  $p(z)$  on the infinitely many sheeted Riemann surfaces which are cut along the negative half of real axis.

Applying Lemma 1, Nunokawa [2] obtained Theorem C.

**Theorem C.** Let  $p(z) \in \mathcal{N}$ ,  $p(z) \neq 0$  in  $\mathbb{E}$  and suppose that

$$\left| \arg \left( p(z) + \frac{zp'(z)}{p(z)} \right) \right| < \frac{\pi}{2} \alpha(\beta) \quad \text{in } \mathbb{E}$$

where  $0 < \beta \leq 1$ ,

$$\alpha(\beta) = \beta + \frac{2}{\pi} \tan^{-1} \frac{\beta q(\beta) \sin \frac{\pi}{2} (1 - \beta)}{p(\beta) + \beta q(\beta) \cos \frac{\pi}{2} (1 - \beta)},$$

$$p(\beta) = (1 + \beta)^{(1+\beta)/2} \quad \text{and} \quad q(\beta) = (1 - \beta)^{(\beta-1)/2}.$$

Then we have

$$|\arg p(z)| < \frac{\pi}{2} \beta \quad \text{in } \mathbb{E}.$$

**Remark 2.** Theorem C holds to be true for all the case  $0 < \beta$  if we also consider it like as Remark 1.

## ON THE ARGUMENT INEQUALITY OF ANALYTIC FUNCTIONS

In the distortion theorem of analytic function theory, if we suppose some assumptions for  $|f'(z)|$ , then we can easily get some results for  $|f(z)|$  by applying integral inequality

$$|f(z) - f(0)| \leq \int_0^z |f'(t)| |dt|.$$

On the other hand, we can not find out any results for the rotation theorem of analytic functions between  $|\arg p'(z)|$  and  $|\arg p(z)|$ .

## 2. MAIN RESULT.

**Theorem.** Let  $p(z) \in \mathcal{N}$ ,  $p(z) \neq 0$  in  $\mathbb{E}$  and suppose that

$$|\arg p'(z)| < \frac{\pi}{4}(\alpha - 1) \quad \text{in } \mathbb{E},$$

where  $1 < \alpha < 2$ . Then we have

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{in } \mathbb{E}.$$

*Proof.* Let us suppose that if there exists a point  $z_0 \in \mathbb{E}$  such that

$$|\arg p(z)| < \frac{\pi}{2}\alpha \quad \text{for } |z| < |z_0|$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}\alpha.$$

Applying Lemma 1, let us suppose  $\arg p(z_0) = \pi\alpha/2$ , then we have

$$\begin{aligned} p'(z_0) &= \frac{p(z_0)}{z_0} i\alpha k \\ &= \left( \frac{p(z_0) - 1}{z_0} \right) \left( \frac{i\alpha k p(z_0)}{p(z_0) - 1} \right) \\ &= \left( \frac{1}{z_0} \int_0^{z_0} p'(t) dt \right) \left( \frac{i\alpha k p(z_0)}{p(z_0) - 1} \right) \\ &= \left( \frac{1}{r} \int_0^r p'(\rho e^{i\theta}) d\rho \right) \left( \frac{i\alpha k p(z_0)}{p(z_0) - 1} \right) \end{aligned}$$

where  $z_0 = re^{i\theta}$ ,  $t = \rho e^{i\theta}$  and  $0 \leq \rho \leq r$ . Therefore we have

$$\arg p'(z_0) = \frac{\pi}{2} + \frac{\pi}{2}\alpha + \arg \left( \frac{1}{r} \int_0^r p'(\rho e^{i\theta}) d\rho \right) + \arg \left( \frac{\overline{p(z_0)} - 1}{|p(z_0) - 1|^2} \right).$$

Applying the property of integral mean of the integral (See Pommerenke [3, Lemma 1]), we have

$$\begin{aligned} \arg p'(z_0) &\geq \frac{\pi}{2} + \frac{\pi}{2}\alpha - \frac{\pi}{4}(\alpha - 1) - \pi \\ &= \frac{\pi}{4}(\alpha - 1). \end{aligned}$$

This contradicts the assumption.

M. NUNOKAWA, N. TAKAHASHI AND A. OGINO

For the case  $\arg p(z_0) = -\pi\alpha/2$ , we also have the following

$$\arg p'(z_0) = \arg \frac{p(z_0)}{z_0}(-i\alpha k) = \arg \left( \frac{1}{z_0} \int_0^{z_0} p'(t) dt \right) + \arg \left( \frac{-i\alpha k p(z_0)}{p(z_0) - 1} \right)$$

where  $1 \leq k$ . Therefore, we have

$$\begin{aligned} \arg p'(z_0) &= -\frac{\pi}{2} - \frac{\pi}{2}\alpha + \arg \left( \frac{1}{r} \int_0^r p'(\rho e^{i\theta}) d\rho \right) + \arg \left( \frac{\overline{p(z_0)} - 1}{|p(z_0) - 1|^2} \right) \\ &\geq -\frac{\pi}{2} - \frac{\pi}{2}\alpha + \frac{\pi}{4}(\alpha - 1) + \pi \\ &= -\frac{\pi}{4}(\alpha - 1). \end{aligned}$$

This contradicts the assumption and so this completes the proof.  $\square$

**Acknowledgement.** We sincerely thank the chance to do this research at Kyoto University which was given by the Research Institute for Mathematical Sciences, Kyoto University from Jan. 7, 2002 to Jan. 9, 2002.

#### REFERENCES

- [1] S.S. Miller and P.T. Mocanu, *Marx-Shrohhäcker differential subordination systems*, Proc. Amer. Math. Soc., **99**(3), (1987), 527-534.
- [2] M. Nunokawa, *On the order of strongly starlikeness of strongly convex functions*, Proc. Japan. Acad., **69**(7), Ser.A (1993) 234-237.
- [3] Ch. Pommerenke, *On close-to-convex analytic functions*, Trans. Amer. Math. Soc., **114**(1), (1965) 176-186.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GUNMA, ARAMAKI MAEBASHI GUNMA 371-8510, JAPAN

E-mail:

Mamoru Nunokawa: nunokawa@edu.gunma-u.ac.jp

Norihiro Takahashi: norihiro@math.edu.gunma-u.ac.jp

Akihisa Ogino: ogino@math.edu.gunma-u.ac.jp